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Lattice Basis Reduction Attack on Matrix NTRU

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Abstract NTRU is one of the most important post-quantum cryptosystems nowadays and since its introduction several variants have been proposed in the literature. In particular, the Matrix NTRU is a variant which replaces the NTRU polynomials by integer matrices. In this work, we develop a lattice-based reduction attack on the Matrix NTRU cryptosystem that allows us to recover the plaintext. We also show that this system is completely vulnerable to the proposed attack for parameters that could be used in practice. We show that this practical attack can also be extended by reducing the lattice dimension. In addition, we give sufficient conditions to avoid decryption failure for the Matrix NTRU.

Keywords: Post-quantum cryptography, NTRU, Matrix NTRU, Lattice cryptanalysis, Lattice attack, Decryption failure, Cryptanalysis

1 Introduction

Quantum computers are now a reality and if large-scale ones are built, they could break most of the public key cryptosystems used currently. That is one of the reasons that research on post-quantum systems (cryptographic systems underpinned on mathematical problems that are intractable by both quantum and conventional computers) has taken off in the last decades.

In 2016, the National Institute of Standards and Technology (NIST) started a public process for the standardization of post quantum algorithms that could be used to interoperate with existing communications protocols and networks currently in use. One of the main mathematical problem underpinning such system is the shortest vector problem (SVP), which aims at finding a shortest vector in a structure called lattice (see Silverman *et al*. [2008]). Two lattice-based systems have progressed significantly in the NIST Post-Quantum Cryptography standardization process, namely NTRU Chen *et al*. [2020a] and NTRU Prime Bernstein *et al*. [2017], both based on the original NTRU system from Hoffstein *et al*. [1998].

The NTRU system was presented in 1996 to the cryptographic community during the CRYPTO'96 rump session. Since its introduction, the NTRU system has been extensively analyzed and extended in many different ways. Interesting review articles containing a wealth of examples and references include Singh and Padhye [2016]; Salleh and Kamarulhaili [2020] and Mittal and Ramkumar [2022]. NTRU is based on modular algebra of polynomials in specific rings and such generalizations range from changing the polynomial coefficients to using matrix structures.

The first NTRU generalization using matrices was introduced in Coglianese and Goi [2005], where key generation, encryption and decryption operate over matrices whose entries are polynomials. In 2008, Nayak *et al*. [2008] proposed a matrix-based system as a variant of NTRU using only matrices with integer entries, replacing the arithmetic of truncated polynomials with a simple modular arithmetic on matrices. This new system, called Matrix NTRU, was subject to some further analysis and even suggested for real data applications. Indeed, Kumar *et al*. [2013] presented a framework for deploying Matrix NTRU in real data applications. More recently, Wijayanti *et al*. [2023a] developed a variant of the Matrix NTRU which is based on matrix arithmetic defined over $\mathbb{Z}[\sqrt{-3}]$ and do Rêgo Sousa and Neto [2025] improved the key generation and corrected the decryption failure bound.

1.1 Related works

Since its introduction the Matrix NTRU has attracted some attention. A comparative study regarding speed and key sizes was performed in Nayak *et al*. [2012] showing that Matrix NTRU was faster than the classical NTRU for short dimensions. In addition, Nayak *et al*. [2012] argue that Matrix NTRU was more secure than the classical NTRU since matrices are non-commutative. However, we will show that the Matrix NTRU system is seriously affected by the lattice attack developed in Section 5. This demonstrates that the Matrix NTRU is far weaker than the classical NTRU system for comparable key sizes, contrary to what was accredited in the literature (see Nayak *et al*. [2012] and Kumar *et al*. [2013]).

Subsequent works on the Matrix NTRU focused on expanding the key space and on eliminating restrictions on the matrices representing the public key by using Gram-Schmidt orthogonalization and companion matrices (Luo and Lin [2011]; Tripathi *et al*. [2016]; Mamdikar *et al*. [2013]).

With regards to the system's security, Nayak *et al*. [2011] applied a reaction type attack to Matrix NTRU, by adapting the results of the original attack presented in Hall *et al*. [1999] for the NTRU system. In addition, a meet in the middle attack was recently presented in Wijayanti *et al*. [2023b]. However, the simulation studies presented in these works only show the performance of the aforementioned attacks for very low

dimensions of the Matrix NTRU, which cannot be used in practice.

Our main contribution to Matrix NTRU is in developing a lattice-based attack and analyse its resistance in dimensions much higher than the ones suggested for practical applications in Kumar *et al*. [2013]. We will show in Section 5 how this attack can recover the private key (up to a permutation). In addition, this attack is further used to construct a message recovery attack that works for a dimension of the system of up to 110. According to Nayak *et al*. [2012], a Matrix NTRU system such as this is equivalent to an NTRU system using polynomials of degree 110^2, which is remains far from being broken by lattice-based techniques (Chen *et al*. [2020a]). This attack is further improved by using the dimension reduction ideas of May and Silverman [2001] leading to practical attacks for $n = 120$.

In addition to the lattice attack, we also analyze the decryption failure rate of the Matrix NTRU. The results of Proposition 2 give sufficient conditions that depend only on the system's public parameters to avoid decryption failure.

1.2 Structure of the paper

The remainder of the paper is organized as follows. In Section 2, we describe the NTRU cryptosystem in detail, and a brief description of Matrix NTRU is given in Section 3. Sufficient conditions to avoid decryption failure for the Matrix NTRU are also presented in this section. Then, in Section 4, we construct the lattice structure associated with the Matrix NTRU and report the results of an attack to recover the private key. This attack is further refined in Section 5 to construct a message recovery type attack that works for a dimension of up to $n^2 = 110^2$ on a personal computer. Finally, Section 6 improves the attack by using the dimension reduction techniques of May which allowed us to run it successfully for $n = 120$ on a personal computer. The results of the previous section are summarized in Section 7 where we explain why the Matrix NTRU is far weaker than the NTRU cryptosystem concerning the lattice attack.

2 The NTRU Cryptosystem

Let n and p be prime numbers. Let $q \geq 1$ be an integer such that $(n, q) = (p, q) = 1$. The main arithmetic operations in the NTRU cryptosystem are calculations over polynomials defined over the rings R, R_p e R_q defined as:

$$R = \frac{\mathbb{Z}[x]}{(x^n - 1)}, R_p = \frac{(\mathbb{Z}/p\mathbb{Z})[x]}{(x^n - 1)}, R_q = \frac{(\mathbb{Z}/q\mathbb{Z})[x]}{(x^n - 1)}.$$

We can notice that the ring R is related to the other two. In fact, for every $a(x)$ in R, we can identify it to an element in R_p or R_q by applying reduction modulo p or q, respectively, over the coefficients of $a(x)$. We say that $a(x)$ is a ternary polynomial if its coefficients lie in the set $\{-1, 0, 1\}$.

The NTRU cryptosystem works as follows.

1. Key Generation: Alice chooses two ternary polynomials at random, namely, $f(x)$ and $g(x)$. Next, Alice tries to compute the inverse of f over the rings R_q, R_p, i.e, $f_q(x) = f(x)^{-1} \in R_q$ and $f_p(x) = f(x)^{-1} \in R_p$ until she succeeds. Finally, she computes the polynomial
$$h(x) = f_q(x) * g(x) \in R_q,$$
where $*$ denotes polynomial multiplication in R_q, i.e., a cyclic convolution product as defined in [Hoffstein *et al*., 1998, Section 1.1] The polynomial $h(x)$ is Alice's public key and the pair $(f(x), f_p(x))$ is her private key.
2. Encryption: Suppose Bob wants to send an encrypted message to Alice and let $m(x) \in R_p$ be Bob's plaintext. Next, Bob chooses, at random, a ternary polynomial $r(x)$ and computes
$$e(x) \equiv ph(x) * r(x) + m(x) \mod q.$$
Notice that Bob's ciphertext $e(x)$ is an element of the ring R_q.
3. Decryption: Once Alice receives Bob's ciphertext $e(x)$, she starts the decryption process by computing
$$a(x) \equiv f(x) * e(x) \mod q.$$
Then the reduction modulo p gives the desired plaintext
$$b(x) \equiv f_p(x) * a(x) \mod p.$$

Because of the randomness of the polynomial $r(x)$, NTRU is a probabilistic cryptosystem, since a message $m(x)$ can be encrypted to several ciphertexts $ph(x) * r(x) + m(x)$, depending on each instance of $r(x)$. In addition to that, during decryption the coefficients of the polynomial $a(x)$ can not grow deliberately because that would invalidate the computations modulo p that follows, a phenomenon known as a decryption failure. Some attacks in the literature take advantage of these decryption failures Howgrave-Graham *et al*. [2003]; Gama and Nguyen [2007]; Jaulmes and Joux [2000] and therefore we should choose the parameters carefully.

3 The Matrix NTRU Cryptosystem

Let p be a prime number and $q >> p$ an integer such that $(p, q) = 1$. Let

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

be an $n \times n$ matrix whose entries are integer numbers, that is, $A \in M_n(\mathbb{Z})$. In a similar way we have done before with polynomials, we say that A is a ternary matrix if all of its entries lie in the set $\{-1, 0-, 1\}$.

We say that A is reduced modulo p, denoted as $A \mod p$, if every entry of A is reduced modulo p. Therefore

$$A \mod p = \begin{pmatrix} a_{11} \mod p & \dots & a_{1n} \mod p \\ \vdots & \ddots & \vdots \\ a_{n1} \mod p & \dots & a_{nn} \mod p \end{pmatrix}.$$

Notice that, in the latter case, we can view matrix A as an element of the ring $M_n(\mathbb{F}_p)$. In this work, we are going to consider the center lift operation modulo p and q in the

decryption process. In that case, it can be useful to consider a non-canonical representation of the elements on the rings. Therefore, when we are dealing with a prime modulus, we have

$$\mathbb{F}_p = \left\{-\frac{p-1}{2}, \ldots, \frac{p-1}{2}\right\}$$

and for a composite number, we have

$$\mathbb{Z}_q = \left\{-\frac{q}{2}+1, \ldots, \frac{q}{2}\right\}.$$

Modular arithmetic over the rings $M_n(\mathbb{F}_p)$ and $M_n(\mathbb{Z}_q)$ is what underpins the Matrix NTRU system. In such rings we can add, subtract and multiply matrices in the same way we have done with matrices over the field of real numbers. However, in order to invert a matrix in $M_n(\mathbb{F}_p)$ we have to be sure that p and the determinant of A are relatively prime Jacques-García *et al*. [2022]; Wijayanti *et al*. [2023b]. If that is the case, then there exists a (unique) matrix B such that $AB = BA = I$ in $M_n(\mathbb{F}_p)$, where I is the identity matrix. The matrix B is called the inverse of A modulo p and denoted $A^{-1} \mod p$.

The Matrix NTRU system works as follows.

1. Key generation: The private and public keys are $n \times n$ matrices. First, we choose a pair of ternary matrices F, G such that $F \mod p$ and $F \mod q$ are invertible in $M_n(\mathbb{F}_p)$ and $M_n(\mathbb{Z}_q)$, respectively. Then we compute the matrices F_p and F_q, where $F_p = F^{-1} \mod p$ and $F_q = F^{-1} \mod q$. The pair F, F_p is the private key and the parameters F, G, F_p, F_q should be kept in secret. Now, we can compute the public key by performing the calculation
$$H = pF_qG \mod q.$$
2. Encryption: To encrypt a message, we encode it as a matrix $M \in M_n(\mathbb{F}_p)$ and choose a ternary matrix $R \in M_n(\mathbb{Z})$, at random. Now, we compute the ciphertext as
$$E = HR + M \mod q.$$
3. Decryption: To decrypt, we compute
$$A = FE \mod q,$$
where F is the private key we saw previously. Under simple conditions on the system parameters (see Proposition 1 bellow), reducing A modulo p gives the desired plaintext
$$B = F_pA \mod p$$

Depending on the parameters selection, we can have errors when operating on Matrix NTRU, just as we have when dealing with NTRU cryptosystem. Sometimes, the matrices B and M can be different and we say there is a decryption failure if that is the case. We notice that the authors Nayak *et al*. [2008] did not address that question. However, it turns out that this is an important issue concerning any cryptosystem since attackers can take advantage of it. To fill in this gap, we prove the following result, which shows that there are parameter selections to avoid decryption failure on Matrix NTRU.

Proposition 1. *Suppose that p, q and n are fixed parameters for the Matrix NTRU defined above. If $n(3p-1) < q$, then any ciphertext E resulting from the encryption of a message M with private key F, decrypts correctly to M.*

Proof. During decryption, one computes

$$\begin{aligned} A &= FE \mod q \\ &= F(HR+M) \mod q \\ &= FHR + FM \mod q \\ &= pFF_qGR + FM \mod q \\ &= pGR + FM \mod q. \end{aligned} \tag{1}$$

Following the same arguments as in the proof of Proposition 6.48 in Silverman *et al*. [2008], we need to bound the largest coefficient (in modulus) of each entry of the matrix $pGR + FM$ computed exactly (without module q). For decryption to work, the above matrix should have entries whose absolute value does not exceed $q/2$. Since every entry of the matrices G and R lie in $\{-1, 0, 1\}$, the largest possible entry of the product GR is upper bounded by n. On the other hand, the matrix M has coefficients between $-(p-1)/2$ and $(p-1)/2$ and F has coefficients lying in $\{-1, 0, 1\}$. Therefore, all coefficients of the product FM can be bounded by $n(p-1)/2$. Finally, the entries of the matrix $pGR + FM$ are all upper bounded in module by $np + n(p-1)/2 = n(3p-1)/2$. Under the assumption of the proposition, this can be further bounded by $q/2$ which will ensure correct decryption of the ciphertext. □

We implemented the Matrix NTRU in `sagemath` (Team [2024]) and assessed the decryption failure rate for a set of different parameters. Table 1 shows the proportion of ciphertext that could not be decrypted correctly, over 10000 different replications of the process for varying n and q. Under the conditions of Proposition 1, we avoid decryption failure whenever $n(3p-1) < q$, and the results for the parameters satisfying these conditions are shown by a gray strip in Table 1. One important fact about Proposition 1 is that as n grows, q should also grow, otherwise one might have decryption failures in the Matrix NTRU system. A similar situation also happens in the classical NTRU system (see [Silverman *et al*., 2008, Proposition 6.48])

4 Lattice attack on the private key

In the following section we show how the problem of finding the private key F of the Matrix NTRU system is connected to solving a well-known problem in lattices, namely finding a shortest vector in a lattice. A lattice can be defined in the following way.

Definition 1. *([Silverman* et al*., 2008, Section 6.4]) Let $f_1, \ldots, f_n \in \mathbb{R}^n$ be a set of linearly independent vectors. The lattice L generated by $f_1, \ldots, f_n$ is the set of linear combinations of $f_1, \ldots, f_n$ with coefficients in $\mathbb{Z}$,*

$$L = \{\alpha_1 f_1 + \cdots + \alpha_n f_n : \alpha_1, \ldots, \alpha_n \in \mathbb{Z}\}. \tag{2}$$

Table 1. Decryption failure estimated probability for the Matrix NTRU system for fixed $p = 3$ and varying n and q. For each parameter setting a new key and message were generated and encrypted. Results present the proportion of messages decrypted INCORRECTLY over 10 000 repetitions. Gray strips represent experimental decryption failure rate where the conditions of Proposition 1 are met.

	q										
n	32	64	79	128	256	307	512	701	1024	2048	4096
5	0.008	0	0	0	0	0	0	0	0	0	0
10	0.775	0	0	0	0	0	0	0	0	0	0
20	1	0.210	0.008	0	0	0	0	0	0	0	0
30	1	0.989	0.397	0	0	0	0	0	0	0	0
40	1	1	0.989	0.002	0	0	0	0	0	0	0
50	1	1	1	0.039	0	0	0	0	0	0	0
60	1	1	1	0.257	0	0	0	0	0	0	0
70	1	1	1	0.734	0	0	0	0	0	0	0
80	1	1	1	0.984	0	0	0	0	0	0	0
90	1	1	1	1	0	0	0	0	0	0	0
100	1	1	1	1	0	0	0	0	0	0	0
110	1	1	1	1	0	0	0	0	0	0	0

The set of vectors $f_1, \ldots, f_n$ is called the lattice basis and it is usual to stack them into a matrix and work with the matrix as being the lattice basis that generates L. A fundamental problem in lattice is finding a shortest nonzero vector in the lattice which minimizes the Euclidean norm $\|f\|$. This is called the shortest vector problem (SVP). It is important to notice that the SVP problem asks for a shortest vector and not the shortest vector since e.g. f and $-f$ have the same Euclidean norm. According to Silverman *et al*. [2008] finding a solution for the SVP problem can be used to break various cryptosystem, in particular the NTRU system from Section 2 (for certain parameters) and as we will show next how it can be used to finding the private key F in the Matrix NTRU system. It is worth noticing that the current version of NTRU submitted to NISTs post quantum competition Chen *et al*. [2020a] have parameters that avoid private key attacks with current computational resources and it is not practical. On the other hand, the Matrix NTRU variant has a serious vulnerability in its construction, that makes it breakable for quite high values of parameters that could be used in practice.

Proposition 2. *Let F, G be the private keys and let H be the public key of the Matrix NTRU with parameters n, p, q. Let (f_k, g_k) be the k-th line of F and G respectively and p^{-1} be the inverse of p module q. Then, (f_k, g_k) belongs to the lattice generated by the lines of the $2n \times 2n$ block matrix*

$$L = \begin{pmatrix} I_n & p^{-1}H \\ 0_n & qI_n \end{pmatrix}, \tag{3}$$

where I_n is the n dimensional identity matrix and 0_n is the n dimensional zero matrix. In other words, the unknown vector (f_k, g_k) can be written as an integer linear combination of the lines of L (which has only known quantities).

Proof. The public key is defined as $H = pF_qG$, where p and q are coprime, and F and G are the private keys created during the key generation step. Multiplying both sides of the above equation by Fp^{-1} we get $Fp^{-1}H = G \mod q$. This implies that there exists a matrix $A \in M(\mathbb{Z})$ such that

$$G = F(p^{-1}H) + qA, \tag{4}$$

Let $V = (\alpha H)$ and denote by f_k, g_k and a_k the k-th line of the matrices F, G and A. Equation (4) implies that $g_k = f_kV + qa_k$. Therefore, the $1 \times 2n$ vector (f_k, g_k) can be written as

$$\begin{aligned}(f_k, g_k) &= (f_k, f_kV + a_k(qI_n)) \\ &= (f_kI_n + a_k0_n, f_kV + a_k(qI_n)) \\ &= f_k(I_n, V) + a_k(0_n, qI_n) \\ &= f_{k1}(1, 0, \ldots, 0, 0, v_{11}, v_{12}, \ldots, v_{1n}) \\ &+ f_{k2}(0, 1, \ldots, 0, 0, v_{21}, v_{22}, \ldots, v_{2n}) \\ &\ldots \\ &+ f_{kn}(0, 0, \ldots, 0, 1, v_{n1}, v_{n2}, \ldots, v_{nn}) \\ &+ a_{k1}(0, 0, \ldots, 0, 0, q, 0, \ldots, 0) \\ &+ a_{k2}(0, 0, \ldots, 0, 0, 0, q, \ldots, 0) \\ &\ldots \\ &+ a_{kn}(0, 0, \ldots, 0, 0, 0, 0, \ldots, q).\end{aligned} \tag{5}$$

Since all $1 \times 2n$ dimensional vectors at the rhs of (5) equation are exactly the lines of the matrix L in (3), the proof is completed. □

Recall that the matrices F and G have only coefficients in $\{-1, 0, 1\}$, and therefore any line of the type (f_k, g_k) is a relatively short vector in the lattice L for large q. Indeed, using a Gaussian heuristic, the shortest vector expected from a lattice L depends only on the dimension of L and its determinant. Since L is upper triangular, we have $\det(L) = q^n$. Therefore, the length of the shortest vector expected from L is

$$l = \sqrt{\frac{n}{2\pi e}}(\det(L))^{1/(2n)} = \sqrt{\frac{nq}{2\pi e}}.$$

The target vector (f_k, g_k) has varying norm depending on how the private key is chosen. If every entry is chosen with uniform probability on $\{-1, 0, 1\}$, then, its expected norm is $\alpha = \sqrt{2n/3}$. This means that for higher values of q, the target vector has norm smaller than the expected by the by the Gaussian heuristics, which means that the LLL algorithm has a high probability of find a short vector in L (see Silverman *et al*. [2008]).

Another interesting fact is that we do not need to attack the whole private key F which has dimension n^2. Using a suitable algorithm to solve the SVP in L we can try to find any line of F and hopefully all lines separately reducing the complexity of the attack by a factor of n.

Solving the SVP problem can be done in many ways, and one commonly done is by using lattice-based reduction algorithms such as the famous LLL algorithm from Lenstra *et al*. [1982] and or the BKZ algorithms Chen and Nguyen [2011]. For more details on the LLL algorithm see e.g. [Bremner, 2011, Chapter 4]. These base reduction algorithms try to find a shorter basis for the lattice L, and in doing so, they usually return potential short vectors for the lattice L and we can test if they correspond to any line of the matrix F.

In addition to using the LLL algorithm, there is an improved variant of lattice based reduction algorithms called BKZ as defined in Chen and Nguyen [2011]. The BKZ algorithm, proceeds with repeated local improvements to the lattice basis. One of its simple implementations is a recurring set of calls to SVP oracle solvers of dimension set by the block size parameter of the BKZ algorithm and LLL calls and will be used in the following for recovering the private key. Although these algorithms can be further refined to lead to better solutions for the SVP problem, the first vectors of the BKZ output are already short enough to give us candidates for lines of the private key matrix F.

In what follows we use the result of Proposition 2 to recover the private key matrix F in the following way.

Algorithm 1: Recovering private key F:

Let H the public key of the Matrix NTRU system with parameters n, p, q.

1. Compute $p^{-1} \mod q$and create the matrix L from eq.(3).
2. Run the BKZ algorithm on L and get L_{red}.
3. Use L_{red} to create a submatrix L^*_{red} with entries corresponding to the first n columns of L_{red}.
4. For each line f of the matrix F, check whether or not f is contained in one of the lines of L^*_{red}.

We implemented the above algorithm to assess the performance of the attack for several values of the parameters. Namely, we choose $q \in \{256, 4096\}$ for varying n and recorded what proportion of the lines of F one can recover from the attack using only the public key and public parameters p, q, n. We fix $p = 3$, a common choice in NTRU-like systems, as it simplifies modular arithmetic while preserving the essential algebraic structure needed for encryption and decryption. This setting is sufficient to investigate the system's behavior and assess the effectiveness of our attack strategies, since the other parameters are allowed to vary.

As showed in Figures 1 and 2, we see that the attack, in almost all experiments, can recover all lines of the matrix F for $n \in 40, 50, \ldots, 80$ and $q = 256$ and for $n \in 40, 50, \ldots, 110$ and $q = 4096$. In particular, breaking Matrix NTRU for $n = 110$ means breaking a private key that has $n^2 = 12100$ entries (which is equivalent to a huge private key polynomial in the classical NTRU), and the success of our experiments at this scale reveals a potential vulnerability.

For $q = 4096$, we also tested $n = 115$ and observed that the attack succeeded in approximately half of the experiments; however, in the other half, the BKZ algorithm stops the execution, indicating that the reduction process aborted before completion. This is expected, as the lattice dimension at this stage is $2n = 230$, which makes basis reduction more difficult. Although all the results in this work were obtained using a BKZ block size of 10, which already produced promising results, we increased the block size to investigate why BKZ stops the execution.

For $n = 115$, using the default BKZ block size of 10 or 15 caused the reduction to fail as we explained. However, increasing the block size to 20 or 25 resolved these issues, and in both cases the returned basis contained the private key, indicating a successful attack at the cost of increasing the running time (1 minute for block size 10 and 8 minutes for block size 25).

These results confirm that the attack remains feasible for higher dimensions, albeit at the cost of increasing the BKZ block size. Since the complexity of BKZ is exponential in the block size, the running time of the attack grows rapidly as the parameter increases [Chen *et al*., 2020b, Section 6.3].

One issue with the attack of algorithm 1 is that even though we can recover some lines of the matrix F, we still do not know how to reorder them to reconstruct the true F and use it for decryption. Nevertheless, we will see in the next Session that any permutation of lines of the private key F can be used to successfully decrypt a message encrypted with F and this allows us to construct a message recovery attack on the Matrix NTRU system.

5 Message recovery attack

In the previous section we showed how to construct the associated lattice for the Matrix NTRU system and demonstrated the viability of the attack for recovering, up to a permutation, the whole private key matrix F. In what follows we show that this attack can be used to successfully decrypt a message encrypted with F, even though the correct order of the lines of F is not known by the attacker.

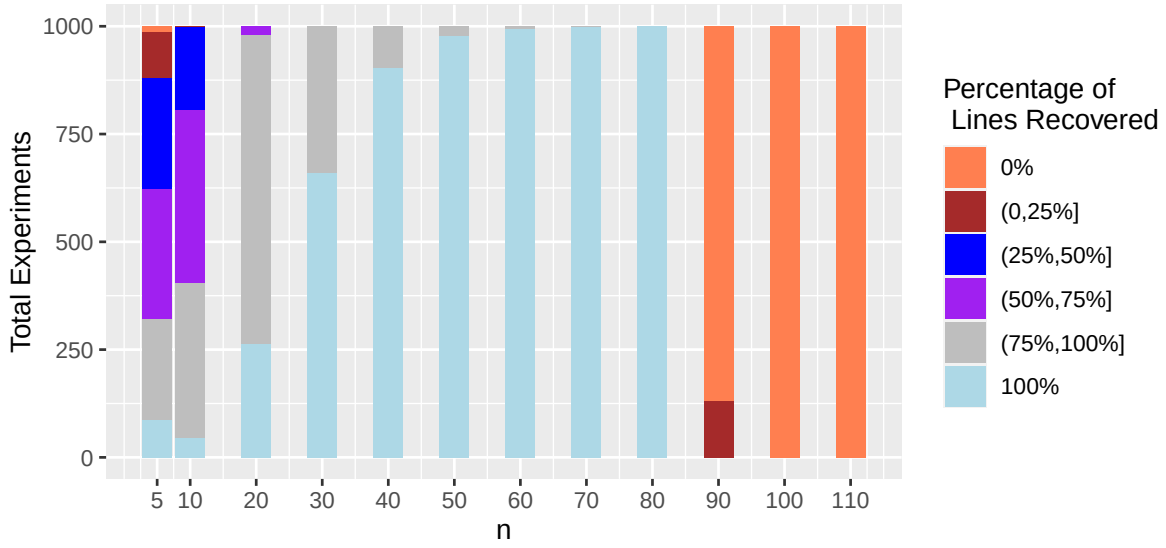
Proposition 3. *Let F, H be a corresponding private and public key pair for the Matrix NTRU system with parameters n, p, q and let E be a ciphertext associated with the encryption of a message M such that the decryption process of E using F correctly returns M. Let F^* be a matrix formed by permuting the lines of the matrix F. Then, applying the decryption process to E using F^* returns the message M.*

Proof. By the construction of F^*, there exists an uni-modular matrix D, such that $F^* = DF$. Applying the decryption process with F^* gives

$$A^* = F^*E \mod q = DF(HR + M) \mod q.$$

Using the definition of the matrix H and the fact that $FF_q = I$ we get,

Figure 1. Performance of the attack of algorithm 1 for recovering lines of the matrix F for the Matrix NTRU with parameters n and fixed $q = 256$. For each experiment, we generate new keys F and H and apply the attack recording how many experiments were able to recover all lines of F (100%), between 1 and $n - 1$ lines of F and none of the lines of F (0%).



$$\begin{aligned} A^* &= DF(HR + M) \mod q \\ &= DF(pF_qGR + M) \mod q \\ &= DpFF_qGR + DFM \mod q \\ &= pDGR + DFM \mod q \end{aligned} \tag{6}$$

The next decryption step is to compute A^* modulo p and notice that since D only change signs and permute lines of GR, we can still make this operation without incurring in the risk of extrapolating the norm of the entries of the matrix DGR. Therefore, we have

$$A^* \mod p = 0 + DFM = F^*M.$$

Finally, using the inverse F_p^* of $F^* \mod p$ we compute

$$C^* = F_p^*A^* = F_p^*F^*M \mod p = M.$$

□

We use the result of Proposition 3 in combination with Algorithm 1 to create a practical message recovery attack. This attack uses the upper left part of the reduced lattice basis returned by the BKZ algorithm as a potential key. As we saw in Section 4, this gives us in several cases the whole private key (up to a permutation of lines). The result of Proposition 3 says this potential key can still be used to decrypt a message successfully and this is tested in practice.

For each experiment, we generate new key pair F, H a new message M, encrypt it and try to decrypt with the key F^* returned from the attack of algorithm 1. The experiment was repeated 100 times on an Intel(R) Core(TM) i5-9500T CPU 2.20GHz and the success rate reported in Table 2. In the worst case scenario tested ($n = 110$ and $q = 4096$) we can recover the true message in less than one minute. For $q = 256$ the attack works for dimension until $n = 80$ and for $q = 4096$ it works for dimensions up to 100. Since Nayak *et al.* [2012] a Matrix NTRU of dimension n with an NTRU of dimension n^2, this clearly shows that Matrix NTRU is far weaker than NTRU since an NTRU with dimension $100^2 = 10\,000$ is far from being broken. In fact, the highest security suggested for NTRU in practical applications has parameters $n = 821$ and $q = 4096$ as shown in [Chen *et al.*, 2020a, Section 1.6]. By the results of this section, the Matrix NTRU of dimension n^2, should be comparable (in terms of the lattice attack) to at most an NTRU of dimension n. Therefore, even though the Matrix NTRU can encode a similar number of bits in the private key as the NTRU, its internal structure makes it vulnerable to attacks in a much smaller dimension. In addition, a direct consequence of Proposition 3 is that every private matrix F of dimension $n \times n$ has $n!$ equivalent keys, i.e., any of them can decrypt correctly. It is worth noticing that in the classical ntru there are n equivalent keys for each private key.

6 Applying May's Idea

As we can see in Figure 1, our attack against Matrix NTRU becomes ineffective as the lattice dimension increases. This is because the execution time and complexity of BKZ algorithm grow exponentially with the lattice dimension.

In 1999, Alexander May May [1999] presented a novel idea on the cryptanalysis of the NTRU cryptosystem. In his approach, he could work with lattices of smaller dimension compared to earlier attacks on NTRU. This was a key contribution because reducing the lattice dimension can make lattice reduction algorithms like LLL and BKZ more efficient.

Using lattice reduction and taking advantage of the special structure of NTRU secret keys, he was able to cut a certain number of columns in the matrix constructed over the NTRU lattice. These new lattices enabled him to break some medium security instances of NTRU, at that time, in less than 1 hour. Even for higher security levels, he could recover some keys, namely the weak ones.

Figure 2. Same settings as in Figure 1 but with $q = 4096$.

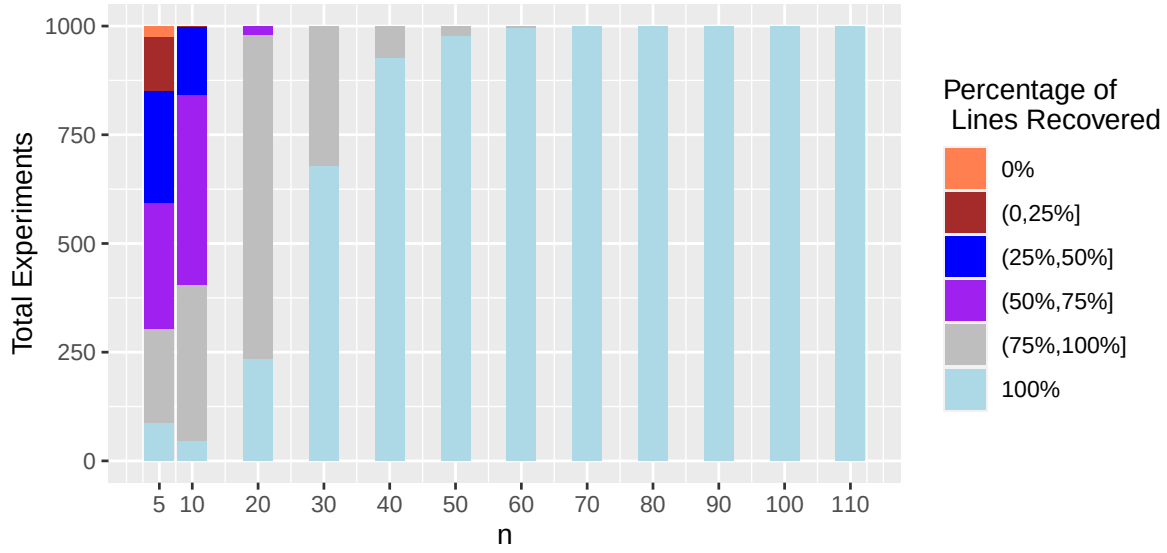


Table 2. Performance of the message recovery attack for the Matrix NTRU with parameters n and fixed $q \in \{256, 4096\}$. For each experiment, we generate new key pair F, H a new message M, encrypt it and try to decrypt with the key F^* returned from the attack of algorithm 1. The experiment was repeated 100 times and the success percentage recorded.

q	$n = 5$	$n = 10$	$n = 20$	$n = 30$	$n = 40$	$n = 50$	$n = 60$	$n = 70$	$n = 80$	$n = 90$	$n = 110$
256	1	1	1	1	1	1	1	1	0	0	0
4096	1	1	1	1	1	1	1	1	1	1	1

As we can see in Silva *et al*. [2024], we can apply May´s technique to recover keys over NTRU variants. That's the case in this section, where we are going to apply May's idea to improve the results of the previous sections.

The idea is quite simple. Instead of using all possible basis vectors, we can select only those most relevant to find short vectors and, thus, calculate the private key on Matrix NTRU cryptosystem. This reduces the computational complexity of lattice reduction algorithms, which enables us to apply BKZ algorithm for larger matrix dimensions.

In this work, as in Silva *et al*. [2024], we will reduce lattice dimension by removing columns, even though this procedure can introduce some inaccuracies in the lattice structure. However, despite potential inaccuracies, the reduced lattice may retain enough information to enable private key recovery.

Columns can be removed randomly, but the attack's effectiveness depends on which columns are discarded. May's strategy aims to lower the lattice dimension while preserving vectors that contain critical information. If the wrong columns are removed, the reduced lattice may lack the necessary properties for a successful attack. In this work, column removal was performed on the right side of the matrix.

Therefore, even after reducing the original lattice dimension, we can recover the private key in the Matrix NTRU cryptosystem. Next, we are going to see the experimental results.

6.1 Experimental results of the lattice attack based on Mays idea

Figure 3 summarizes the results we obtain through the application of May's idea to recover the private key of the Matrix NTRU system. Using the new lattice with smaller dimension we could attack the Matrix NTRU system for dimensions up to 120, which was not possible on a personal computer using out of the box BKZ as was done in Section 4.

To appreciate the impact of May's idea, recall that in Section 4 we applied BKZ to a lattice of dimension $2n$. The attack was successful when we were able to recover the entire private key, that is, when the reduced lattice contained all n rows of the private key. The practical limitation of the attack lies in the required computational resources, which constrained our experiments to $n = 110$ (a setting where both the attack and BKZ consistently succeeded on a personal computer). May's approach offers two key advantages:

1. It extends the range of parameters for which the attack is effective (since for a fixed n, the attack can be executed by applying BKZ to a lattice of smaller dimension $2n-k$ for $k > 0$);
2. Running BKZ on a lattice of smaller dimension is faster.

The experiment was designed to recover all lines of the private key of the Matrix NTRU system for fixed $p = 3$, $n \in \{110, 115, 120, 125\}$ and a dimension cut $k \in \{0, 1, \ldots, n-1\}$. For each n we generate 100 different public private key pairs and attack the lattice to recover the private key. The upper panel of **Figure 3** reports on the success rate of the attack and the lower panel reports on whether BKZ runs successfully. These results show that using May's idea still gives

Table 3. Success rates for different strategies of column removal when using May's technique.

n	Strategy	$k = 30$	$k = 40$	$k = 50$	$k = 60$	$k = 70$
100	Last Columns	100.00%	99.00%	97.02%	89.11%	0.00%
100	Random Columns	100.00%	99.00%	100.00%	91.09%	0.00%
110	Last Columns	100.00%	95.05%	84.16%	59.41%	0.00%
110	Random Columns	100.00%	94.06%	90.10%	62.38%	0.00%
115	Last Columns	67.33%	51.49%	39.60%	7.92%	0.99%
115	Random Columns	65.35%	57.43%	39.60%	11.88%	0.99%

an attack that works with success rate above 50% even for cut values up to 60.

For $n = 115$, the success rate starts at around 50% when we use the full lattice ($k = 0$). As we increase the value of k, the lower panel of Figure 3 shows that BKZ runs successfully more often and the attack success rate increases up around 80% (k = 17). As k continues to increase, the success rate starts decreasing reaching 0% for a cut $k \geq 71$.

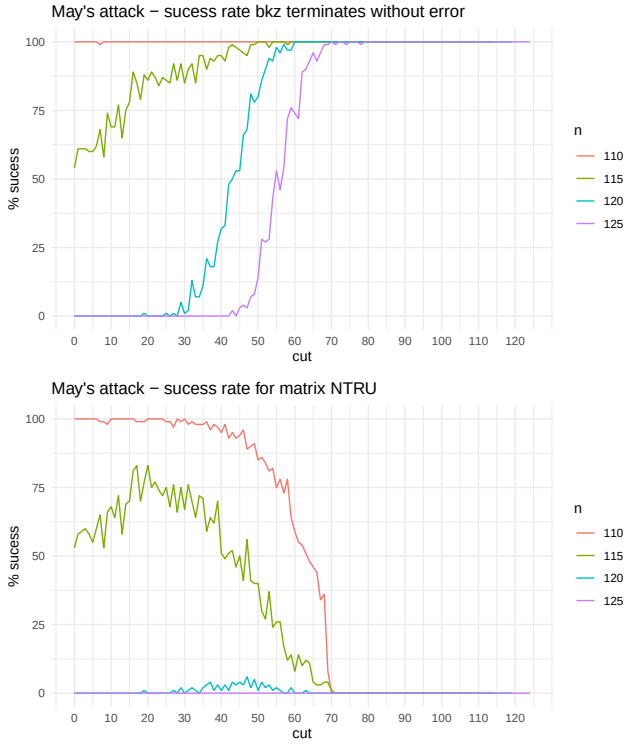
For $n = 120$ the success rate starts at zero due to the fact that BKZ does not run successfully for low values of k as the lattice dimension $240 - k$ is quite high (**Figure 3**, lower panel). As discussed in the previous section, one could increase the block size in BKZ to avoid these errors at the cost of increasing the computational time. For these experiments, we stayed in the fixed scenario where the block size is only 10 as this suffices to show the effectiveness of using May's idea.

Still for $n = 120$, and as the cut value k increases, $k = 19$, one instance out of 100 runs successfully and it indeed contains the private key. As k continues to increase, BKZ starts to run successfully more often and the success rate of the attack increases up to 6% for $k = 47$ and then starts to decrease to zero. For $k \geq 64$, although BKZ runs successfully, the attack does not work as we are cutting too many dimensions of the original lattice and, therefore, we will not be able to recover the private key.

The case $n = 125$ did not return the whole private key and therefore its success rate was zero. Recall that for our attack to work, we need to solve SVP n times to find n lines of the private key matrix. Although in some cases it found a couple of lines of the private key matrix, it did not succeed in finding all lines of the matrix.

One could ask if the way we selected the columns to cut can affect the attack performance significantly. We have already observed at the introduction of Section 6 that the attack performance cannot be significantly improved by changing the strategy on how the columns are removed and this is backed up by an experiment whose results are shown in Table 3. We selected different lattice dimensions and 5 cut values k and repeated the experiment 100 times for each combination of values. Out of the 15 difference scenarios, removing columns at random was slightly better in 6 cases, and its performance was exactly the same at the other scenarios or worse (2 of them).

Figure 3. Lattice attack to recover all lines of the private key of the Matrix NTRU system for fixed $p = 3, q = 4096, n \in \{110, 115, 120, 125\}$ and a dimension **cut** $k \in \{0, 1, \ldots, n-1\}$. For each n we generate 100 different public private key pair and attack the lattice to recover the private key. The upper panel reports on the success rate of the attack and the lower panel reports on whether BKZ runs successfully.



7 Conclusion

In Nayak *et al*. [2012], the authors compare the speed performance for encryption and decryption of the classical NTRU and the Matrix NTRU arguing that Matrix NTRU is faster than NTRU for comparable parameters. They compare a Matrix NTRU with parameters n with an NTRU with parameter n^2, since the original idea of the Matrix NTRU system is to encode the private key polynomial of degree n^2 into a matrix of dimension $n \times n$. In terms of brute force search for the private key they are comparable. In terms of lattice-based attacks, the post-quantum NIST finalist NTRU encrypt (Chen *et al*. [2020a]) with parameters $n = 509$ and $q = 2048$ has already moderate security and cannot be broken by current the available techniques using lattice attacks. The equivalent Matrix NTRU would have dimension around 23, an integer approximation to $\sqrt{509}$. However, a Matrix NTRU with parameters $n = 23$ is completely vulnerable to lattice attacks on a personal computer as showed in Section 4. In fact we can attack such system for even higher dimension.

These results also show that in terms of lattice security, selecting parameters for using Matrix NTRU for a key exchange protocol would require using at least $n = 509$ for a similar security to NTRU encrypt (level 1 security according to NIST specification in 2020). However, it does not make sense to exchange a private key with n^2 entries when there are more efficient post-quantum key exchange systems available for the same security level.

The bottleneck of the proposed attack is in finding a suitable short vector for the Matrix NTRU system using lattice basis reduction algorithms. Therefore, any improvement on these techniques such as replacing the BKZ routine by, e.g., the ones described in (Albrecht and Ducas [2021], May and Silverman [2001], and Bi and Han [2021]; Zhao and Ye [2023]) would improve the results showed in **Figures 1 and 2**.

The lattice attack is inherent to NTRU-type systems and forms the basis for evaluating their security against classical attacks. In our approach, the success of the attack stemmed from exploiting the structural weakness of the private key by targeting its rows independently. A potential countermeasure could involve designing a new variant of matrix-NTRU that incorporates both left and right matrix multiplications during public key generation and encryption/decryption. Such modifications could disrupt the independence of private key components across rows, thereby increasing the diffusion of cryptographic quantities and making attacks more difficult. This direction presents an interesting avenue for future research. However, it is worth noting that matrix multiplication tends to offer inherently lower diffusion compared to polynomial convolution (used e.g. in Bernstein *et al*. [2017]; Tripathi *et al*. [2016]), which may itself pose a security concern.

The vulnerability of the Matrix NTRU system is significant, as the presented lattice attacks effectively reduce its security from n^2 to n. Comparing with the NTRU cryptosystem, the Matrix NTRU tries to create a faster variant of NTRU by separating the private key into slices (lines of a matrix) and using this to create the public key. On the other hand, this come at a very high security cost since just one line is used at a time to construct the lines of the public key. That means it runs faster at the cost of reducing the diffusion during the creation of the public key. On the other hand, NTRU creates public keys by using convolution of polynomials, and therefore, every coefficient contributes to create every coefficient of the public key. In addition, we have increased the attack performance by showing how May's approach of lattice dimension reduction can be applied successfully in the Matrix NTRU system. We show that it is possible to extend the applicability of BKZ algorithm to this system by using a similar approach to that used to attack NTRU.

In this work, we showed that it is possible to decrypt ciphertexts in Matrix NTRU cryptosystem if we just have a matrix whose columns are a permutation of the original columns of the private key. As a future work, inspired by Coppersmith and Shamir [1997], one could investigate the viability of decryption using matrices with small norm, i.e., ternary matrices with approximately the same number of non-zero elements as the private key.

To conclude, we believe that the proposed attack can be adapted to other NTRU like systems relying on matrices operations such as the one based on the Gaussian integers.

Declarations

Authors' Contributions

All authors contributed equally to the conception, design, execution, analysis, and writing of the manuscript. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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Availability of data and materials

The Matrix NTRU system and the attack implementations generated during this study are publicly available at `https://github.com/thiagodoregosousa/matrix-ntru.git`

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