Linial's Conjecture for Arc-spine Digraphs *

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Abstract. *A* path partition *$\mathcal{P}$ of a digraph D is a collection of directed paths such that every vertex belongs to precisely one path. Given a positive integer k, the k-norm of a path partition $\mathcal{P}$ of D is defined as $\sum_{P \in \mathcal{P}} \min\{|P_i|, k\}$. A path partition of a minimum k-norm is called k-optimal and its k-norm is denoted by $\pi_k(D)$. A* stable set *of a digraph D is a subset of pairwise non-adjacent vertices of $V(D)$. Given a positive integer k, we denote by $\alpha_k(D)$ the largest set of vertices of D that can be decomposed into k disjoint stable sets of D. In 1981, Linial conjectured that $\pi_k(D) \leq \alpha_k(D)$ for every digraph. We say that a digraph D is arc-spine if $V(D)$ can be partitioned into two sets X and Y where X is traceable and Y contains at most one arc in $A(D)$. In this paper we show the validity of Linial's Conjecture for arc-spine digraphs.*

1. Introduction

For a digraph D, let $V(D)$ denote its set of vertices and let $A(D)$ denote its set of arcs. Given an arc $a = (u, v) \in A(D)$, we say that u and v are *adjacent*. The set of *neighbors* of a vertex u in D, denoted by $N(u)$, is the set of all vertices that are adjacent to u and distinct from u. In this paper, we consider only digraphs without loops and parallel arcs. A *path* P is a sequence of distinct vertices $(v_1, v_2, \ldots, v_\ell)$ such that for every $i = 1, 2, \ldots, \ell - 1$, $(v_i, v_{i+1}) \in A(D)$. We define the order of a path P, denoted by $|P|$, as the number of its vertices. A *Hamilton path* is a path containing every vertex in $V(D)$. We say that a digraph D is *traceable* if it contains a Hamilton path. A *cycle* C is a sequence of vertices $(v_0, v_1, \ldots, v_\ell)$ such that $(v_i, v_{i+1}) \in A(D)$ for every $i = 0, 1, 2, \ldots, \ell - 1$ and all vertices are distinct except precisely v_0 and v_ℓ which coincide. We say digraph D is *acyclic* if it contains no cycles. A digraph D is *transitive* if whenever $(u, v) \in A(D)$ and $(v, w) \in A(D)$, then $(u, w) \in A(D)$ as well.

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Given a path $P = (v_1, v_2, \ldots, v_\ell)$, we denote by *ter(P)* the terminal vertex v_ℓ of P. The subpath $(v_1, v_2, \ldots, v_i)$ of P is denoted by Pv_i, the subpath $(v_i, v_{i+1}, \ldots, v_\ell)$ of P is denoted by v_iP and the subpath $(v_i, v_{i+1}, \ldots, v_j)$ of P is denoted by v_iPv_j. We denote by $\lambda(D)$ the order of a longest path in D. Given another path $Q = (w_1, w_2, \ldots, w_f)$, where $v_\ell = w_1$, we denote the concatenation of P and Q by $P \circ Q = (v_1, v_2, \ldots, v_\ell = w_1, w_2, \ldots, w_f)$.

A *path partition* $\mathcal{P}$ of a digraph D is a collection of directed paths such that every vertex belongs to precisely one path and we denote by $|\mathcal{P}|$ the number of paths in the partition. We say that a path partition $\mathcal{P}$ in D is *optimal* if there is no path partition $\mathcal{P}'$ such that $|\mathcal{P}'| < |\mathcal{P}|$. We denote by $\pi(D)$ the cardinality of an optimal path partition. Given a positive integer k, the *k-norm* of a path partition $\mathcal{P}$ of D is defined as $\sum_{P \in \mathcal{P}} \min\{|P_i|, k\}$. A path partition of D with minimum k-norm is called *k-optimal* and its k-norm is denoted by $\pi_k(D)$. Note that $\pi(D) = \pi_1(D)$.

A *stable set* S in D is a subset of vertices of $V(D)$ such that every two vertices of S are nonadjacent. A stable set with maximum cardinality is called a *maximum stable set* and its cardinality is denoted by $\alpha(D)$. Let k be a positive integer and D be a digraph. A *k-partial coloring* $\mathcal{C}^k$ of D is a set of k disjoint stable sets. Each such stable set is called a *color class*. Note that some vertices may not belong to any of the k color classes. The *weight* of a k-partial coloring is defined as $\sum_{C \in \mathcal{C}^k} |C|$ and it is denoted by $||\mathcal{C}^k||$. We say that $\mathcal{C}^k$ is an *optimal k-partial coloring* of D if it is a partial coloring of maximum weight and we denote its weight by denoted $\alpha_k(D)$. Note that $\alpha(D) = \alpha_1(D)$.

Dilworth [Dilworth 1950] was the first to associate a path partition with stable set in digraphs. In 1950, he showed that $\pi(D) = \alpha(D)$ when the digraph D is transitive and acyclic. A decade later, in 1960, Gallai and Milgram [Gallai and Milgram 1960] generalized Dilworth's Theorem to arbitrary digraphs by relaxing the equality to the inequality $\pi(D) \leq \alpha(D)$. Note that equality does not always hold; for instance if D is a cycle of order 5, then $\pi(D) = 1$ but $\alpha(D) = 2$. Much later, in 1976, Greene and Kleitman [Greene and Kleitman 1976] found a different way to generalize Dilworth's theorem by establishing a relation between $\pi_k(D)$ and $\alpha_k(D)$, i.e., changing the notion of minimality of a path partition and allowing the use of up to k disjoint stable sets to cover the maximum number of vertices possible. They showed that for any transitive acyclic digraph D and any positive integer k, we have $\pi_k(D) = \alpha_k(D)$.

As much as Gallai-Milgram's Theorem extends Dilworth's Theorem by relaxing the equality when dealing with arbitrary digraphs, we may think that the next step is to relax the equality proved by Greene and Kleitman to the inequality $\pi_k(D) \leq \alpha_k(D)$ in order to generalize their theorem for arbitrary digraphs. However, it is an open problem whether such generalization holds. Such question was raised by Linial [Linial 1981] in 1981 and is known as Linial's Conjecture. Some particular cases of the conjecture were already proved. We highlight the cases $k = 1$ (Gallai-Milgram's Theorem itself), $k = 2$ [Berger and Hartman 2008], acyclic digraphs [Linial 1981], bipartite digraphs [Berge 1982], digraphs with $\lambda(D) \leq k$ [Berge 1982] and traceable digraphs [Berge 1982]. For more details on the state of the art of Linial's Conjecture we refer the reader to [Sambinelli 2018, Table 6.1].

There is one recent partial result on Linial's Conjecture that is particularly relevant to this work. In 2017, Sambinelli, Nunes da Silva and Lee proved Linial's Conjecture for a class of digraphs called spine digraphs [Sambinelli et al. 2017]. A digraph D is a *spine digraph* if there exists a partition $\{X, Y\}$ of $V(D)$ such that $D[X]$ is traceable and Y is a stable set in D. Spine digraphs are a superclass of split digraphs. Long before, in 1994, Hartman, Saleh and Hershkowitz [Hartman et al. 1994] gave a proof of a different (although related) conjecture of Linial which we refer to as Linial's Dual Conjecture (see [Sambinelli 2018] for its statement). The proof of Sambinelli, Nunes da Silva and Lee [Sambinelli et al. 2017] has some similarity in structure to that of Hartman, Saleh and Hershkowitz; however some particular technique had to be developed to address Linial's Conjecture. The recent discovery of this new technique motived this work. Here we present an extension of the use of that technique on a superclass of spine digraphs. The purpose of the work is to start the investigation of more superclasses where the new technique may be applied to solve Linial's Conjecture. We started with the class of arc-spine digraphs, defined in the next section.

2. Arc-spine Digraphs

We say that a digraph D is an *arc-spine digraph* if there exists a partition $\{X, Y\}$ in $V(D)$ where $D[X]$ is traceable and $D[Y]$ contains at most one arc. One such partition $\{X, Y\}$ of an arc-spine digraph is *maximal* if X is maximal. We use $D[X, Y]$ to denote that D has one such partition $\{X, Y\}$ and that it is maximal. We denote by a the unique arc of $D[Y]$ that may exist and by u and v the tail and head of a, respectively. Let $P = (x_1, x_2, \ldots, x_\ell)$ be a Hamilton path of $D[X]$.

The proof of Linial's Conjecture for spine digraphs presented by Sambinelli, Nunes da Silva and Lee [Sambinelli et al. 2017] involves defining a canonical path partition and a canonical partial k-coloring that have k-norm and weight differing by exactly one. Then, a subclass of spine digraphs whose optimal partial k-coloring has weight higher than that of the canonical one is identified. Those are called *k-loose spine digraphs*. For the remaining digraphs, called *k-tight spine digraphs*, it is shown that the canonical path partition is not k-optimal. The proof of Linial's Conjecture for arc-spine digraphs presented in this paper follows the same structure while adapting the definitions and arguments for the superclass of arc-spine digraphs. We assume that positive integer k is at least 2 throughout the paper; this is fundamental in many steps of our proof. However, since Gallai-Milgram's Theorem is a proof of Linial's Conjecture for the case $k = 1$, Linial's Conjecture does hold for every positive integer k for arc-spine digraphs.

Let $D[X, Y]$ be an arc-spine digraph. We define a *canonical (path) partition* of $D[X, Y]$ with respect to some Hamilton path P of $D[X]$ as $\{P, (u, v)\} \cup \{(y) \colon y \in Y - \{u, v\}\}$. Clearly, this path partition has k-norm equal to $\min\{|X|, k\} + |Y|$. Hence, $\pi_k(D) \leq \min\{|X|, k\} + |Y|$. We say that P is *zigzag-free* in D if none of the following types of arcs exist in D: (i) (y, x_1) or (ii) (x_ℓ, y), where $y \in Y$; (iii) (v, x_2) or (iv) $(x_{\ell-1}, u)$; (v) (x_{i-1}, u) and (v, x_{i+1}) simultaneously or (vi) (x_i, u) and (v, x_{i+1}) simultaneously or (vii) (x_i, y) and (y, x_{i+1}) simultaneously, where $1 < i < \ell$ and $y \in Y$. The motivation for defining this concept is because if P is not zigzag-free, then there is $X' \supset X$ such that $D[X']$ is traceable and $Y' = V(D) - X'$ induces at most one arc. We ommitt the details on how to obtain X' in each case since it is not had to verify, and state this conclusion as a proposition below.

Proposition 1. *Let $D[X, Y]$ be an arc-spine digraph and let $P = (x_1, x_2, \ldots, x_\ell)$ be a Hamilton path of $D[X]$. Then, P is zigzag-free.*

All of the following properties hold when P is zigzag-free:

Lemma 1. *Let $D[X, Y]$ be an arc-spine digraph and let $P = (x_1, x_2, \ldots, x_\ell)$ be a Hamilton zigzag-free path of $D[X]$. For every subpath $x_q P x_r = (x_q, x_{q+1}, \ldots, x_r)$ of P, if there exists a vertex $y \in Y$ such that y is adjacent to all vertices of $x_q P x_r$, then for every $q \leq i \leq r$:*

(i) $(x_i, y) \in A(D)$ *if* $(x_q, y) \in A(D)$ *and,*
(ii) $(y, x_i) \in A(D)$ *if* $(y, x_r) \in A(D)$.

Proof. The proof is by induction on i. Consider first the case (i); the base case $i = q$ is thus verified. Now, suppose that $i > q$. By the induction hypothesis, we have that $(x_t, y) \in A(D)$ for $q \leq t \leq i - 1$. If $(y, x_i) \in A(D)$, then P is not zigzag-free in D; whence $(x_i, y) \in A(D)$. In particular, when $i = r$, we have shown that $(x_i, y) \in A(D)$ for $q \leq i \leq r$. A symmetric reasoning can be used to prove case (ii). □

Corollary 1. *Let $D[X, Y]$ be an arc-spine digraph and let $P = (x_1, x_2, \ldots, x_\ell)$ be a zigzag-free Hamilton path of $D[X]$. Then there is no vertex $y \in Y$ adjacent to all vertices of P.*

Proof. Assume to the contrary that there is some vertex $y \in Y$ adjacent to every vertex of P. Since P is zigzag-free, $(x_1, y) \in A(D)$. Then, by Lemma 1, $(x_\ell, y) \in A(D)$; a contradiction to the fact that P is zigzag-free. □

Linial's Conjecture is valid for spine digraphs, thus we will assume that $D[Y]$ contains an arc (u, v). Thus, by Corollary 1, there is some vertex $x_v \in X$ that is not adjacent to vertex v. Now let S be any subset of $X - x_v$ containing $\min\{|X| - 1, k - 2\}$ vertices. We may thus define a *canonical k-partial coloring* with respect to S as $\{Y - v, \{v, x_v\}\} \cup \{\{x\} : x \in S\}\}$. Clearly, this k-partial coloring has weight $|Y| - 1 + 2 + \min\{|X| - 1, k - 2\}$. But $\min\{|X| - 1, k - 2\} = \min\{|X|, k - 1\} - 1$; whence $\alpha_k(D) \geq |Y| + \min\{|X|, k - 1\}$ for every arc-spine digraph D.

An arc-spine digraph is *k-loose* if either $|X| < k$ or there is a subset $S \subseteq X$ with $|S| = k$ such that no vertex $y \in Y$ is adjacent to every vertex of S and there are at least two distinct vertices x_u and x_v in S such that $\{u, x_u\}$ and $\{v, x_v\}$ are independent sets. In constrast, an arc-spine digraph is *k-tight* if it is not k-loose, i.e., $|X| \geq k$ and for every subset $S \subseteq X$ with $|S| = k$ either:

(a) there exists $y \in Y : S \subseteq N(y)$ or
(b) there exists $x \in S$ such that $N(u) \cap S = N(v) \cap S = S - \{x\}$.

In Lemma 2 we show that there is an analogue of [Sambinelli et al. 2017, Lemma 1] for k-loose arc-spine digraphs. Note, however, that the concept of k-loose for arc-spine digraphs presented here is different from the concept of k-loose for spine digraphs presented in [Sambinelli et al. 2017]. The different definition of k-loose was needed as a means to guarantee that there would be perfect analogues of Lemmas 1 and 3 from [Sambinelli et al. 2017] for arc-spine digraphs. The analogue of [Sambinelli et al. 2017, Lemma 3] is Lemma 6.

Lemma 2. *If $D[X,Y]$ is a k-loose arc-spine digraph, then:*

(i) $\alpha_k(D) \geq |Y| + \min\{|X|, k\}$ *and*
(ii) $\pi_k(D) \leq \alpha_k(D)$.

Proof. Recall that $\pi_k(D) \leq |Y| + \min\{|X|, k\}$ since this is the k-norm of the canonical partition (even when D is spine there is a path partition with such k-norm). Also, recall that the canonical k-partial coloring $|Y| + \min\{|X|, k-1\}$ (even when D is spine there is a k-partial coloring with such weight). If $|X| < k$, then $\min\{|X|, k-1\} = \min\{|X|, k\}$ in this case and both (i) and (ii) hold. Thus, we may assume that $|X| \geq k$. So, there exists $S \subseteq X$ with $|S| = k$ such that no vertex $y \in Y$ is adjacent to every vertex in S and there are at least two distinct vertices x_u and x_v in S such that $\{u, x_u\}$ and $\{v, x_v\}$ are independent sets. Suppose that $S = \{x_1, x_2, \ldots, x_k\}$ and let $\mathcal{C}_0^k = \{C_1, C_2, \ldots, C_k\}$ be a k-partial coloring in which $C_p = \{x_p\}$, $p = 1, 2, \ldots, k$. By the choice of S, $\{u, x_u\}$ is an independent set for some $x_u \in S$ and $\{v, x_v\}$ is an independent set for some $x_v \in S$ such that $x_u \neq x_v$. We thus add u to the color class of x_u, v to the color class of x_v and every other vertex $y \in Y - \{u, v\}$ to some color class C_p such that $\{y, x_p\}$ is an independent set (which exists by the choice of S). The k-partial coloring $\mathcal{C}^k$ thus obtained has weight $|Y| + k = |Y| + \min\{|X|, k\}$. Therefore, $\alpha_k(D) \geq ||\mathcal{C}^k|| = |Y| + \min\{|X|, k\}$. Hence, we establish that (i) and (ii) hold. This finishes the proof. □

Lemma 3. *Given a k-tight arc-spine digraph $D[X,Y]$, and a zigzag-free path $P = (x_1, x_2, \ldots, x_\ell)$ of $D[X]$, there is an arc $(x_j, y) \in A(D)$ such that $y \in Y$ for some $k - 1 \leq j \leq \ell$.*

Proof. Consider $S = \{x_1, x_2, \ldots, x_k\}$, the set of the k first vertices of P. First assume that condition (a) in the definition of k-tight digraphs is valid, that is, there exists $y \in Y$ such that $S \subseteq N(y)$. Since P is zigzag-free, we have $(x_1, y) \in A(D)$. So by Lemma 1(i), the result follows with $j = k$. We may thus assume that exclusively condition (b) is valid. Let x_t be the only vertex of S not adjacent to both u and v (note that $1 \leq t \leq k$). If $t \neq 1$, we conclude that $(x_1, u) \in A(D)$ and $(x_1, v) \in A(D)$ as P is zigzag-free. By Lemma 1 applied to subpath $x_1 P x_{t-1}$ and u, we conclude that $(x_{t-1}, u) \in A(D)$. If $t = k$, the result follows; we may thus assume that $t < k$. Thus x_{t+1} is in S and v is adjacent to x_{t+1}. Arc $(v, x_{t+1}) \notin A(D)$, otherwise there would be a zigzag in P. The latter assertion is true even if $t = 1$; in fact, the only difference when $t = 1$ is that we do not know the orientation of the arcs joining vertices of S to u. Since $(x_{t+1}, v) \in A(D)$ whenever $1 \leq t < k$, by Lemma 1 applied to subpath $x_{t+1} P x_k$ and v, $(x_k, v) \in A(D)$ and the result follows. □

Lemma 4. *Given a k-tight arc-spine digraph $D[X,Y]$, and a zigzag-free path $P = (x_1, x_2, \ldots, x_\ell)$ of $D[X]$, there is an arc $(y, x_i) \in A(D)$ such that $y \in Y$ for some $1 \leq i \leq \ell$.*

Proof. Consider $S = \{x_{\ell-k-1}, x_{\ell-k}, \ldots, x_\ell\}$, the set of the k last vertices of P. First assume that condition (a) is valid. By Lemma 1, the result follows immediately. We may thus assume that exclusively condition (b) is valid. So let x be the unique vertex of S not adjacent to u and v. Consider first the case in which vertex $x \neq x_\ell$. Then, as u is adjacent to every vertex of $S - \{x\}$, it is adjacent to x_ℓ. Since P is zigzag-free, $(u, x_\ell) \in A(D)$

in this case. Now, when $x = x_\ell$, vertex u is adjacent to every vertex of $S - \{x\}$, it is adjacent to $x_{\ell-1}$. Since P is zigzag-free, $(u, x_{\ell-1}) \in A(D)$ and the result follows in both cases. □

Lemma 5. *Given a 2-tight arc-spine digraph $D[X, Y]$, and a zigzag-free path $P = (x_1, x_2, \ldots, x_\ell)$ of $D[X]$, for each vertex x_p of P, there must be some vertex $y_p \in Y$ adjacent to x_p, for every $p = 1, 2, \ldots, \ell$.*

Proof. Assume, to the contrary, that there is some p such that no vertex of Y is adjacent to x_p. First we show that both u and v are adjacent to every vertex in $X - \{x_p\}$, let $S_q = \{x_p, x_q\}$ be a set of two vertices of X for some $q \neq p$. Since no vertex of Y is adjacent to x_p and D is 2-tight, we conclude (b) holds for S_q. Thus, both u and v are adjacent to x_q. Such argument holds for every q, $1 \leq q \leq \ell$, $p \neq q$. We therefore conclude that both u and v are adjacent to every vertex in $X - x_p$.

Consider first the case in which $p = 1$. As the digraph D is 2-tight, $|X| = \ell \geq 2$. Since P is zigzag-free, $(u, x_\ell) \in A(D)$ and $(v, x_\ell) \in A(D)$. By Lemma 1, we deduce $(v, x_2) \in A(D)$. But then P has a zigzag, a contradiction. We may thus assume $p \neq 1$. By a symmetric reasoning, we can deduce $p \neq \ell$.

Since P is zigzag-free, we conclude that $(x_1, u) \in A(D)$, $(x_1, v) \in A(D)$, $(u, x_\ell) \in A(D)$ and $(v, x_\ell) \in A(D)$. By Lemma 1, we conclude that $(x_q, u) \in A(D)$, $(x_q, v) \in A(D)$ for $1 \leq q < p$ and $(u, x_q) \in A(D)$ and $(v, x_q) \in A(D)$ for $p < q \leq \ell$. But then (x_{p-1}, u, v, x_{p+1}) is a zigzag in P; again a contradiction. □

The most ingenious part of the proof presented by Sambinelli, Nunes da Silva and Lee is the proof of [Sambinelli et al. 2017, Lemma 3] . Next, we show the proof of its analogue for arc-spine digraphs, Lemma 6. Again, note that the concept of k-tight for spine digraphs is different from the presented definition of k-tight arc-spine digraphs. Furthermore, we point out that the proof of the base case of Lemma 6 is considerably more intricate than the base case of [Sambinelli et al. 2017, Lemma 3].

Lemma 6. *Let $D[X, Y]$ be a k-tight arc-spine digraph and let $P = (x_1, x_2, \ldots, x_\ell)$ be a zigzag-free path of D. Then, there exist paths P_1 and P_2 such that:*

(i) $V(P_1) \cap V(P_2) = \varnothing$;
(ii) $|P_1| + |P_2| \geq |X| + k + 1$;
(iii) $ter(P_1) \cup ter(P_2) = \{x_\ell, y\}$, *for some* $y \in Y$;
(iv) $X \subseteq V(P_1) \cup V(P_2)$.

Proof. The proof is by induction on k. The base case is $k = 2$. Given a 2-tight arc-spine digraph D, by Lemma 3 there is a vertex $y \in Y$ such that $(x_j, y) \in A(D)$ for some $x_j \in V(P)$. Among all arcs $(x_j, y_j) \in A(D)$ with $y_j \in Y$ choose an arc a_j such that j is maximum. As P is zigzag-free in D, we have that $j < \ell$ and so the vertex x_{j+1} exists. Note that by Lemma 5 and the choice of a_j, we can claim the existence of some vertex $y_{j+1} \in Y$ such that $(y_{j+1}, x_{j+1}) \in A(D)$.

Claim 1. *Suppose $k = 2$ and let $t < \ell$. If there exist y_t and y_{t+1} such that $(x_t, y_t) \in A(D)$ and $(y_{t+1}, x_{t+1}) \in A(D)$ then:*

- $y_t \neq y_{t+1}$;

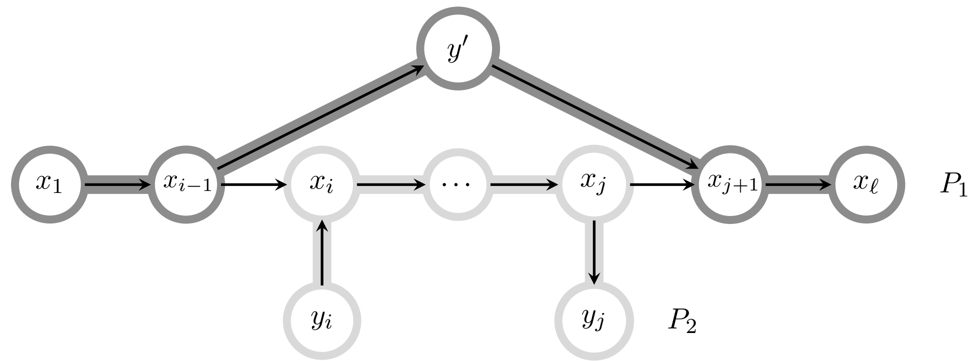

Figure 1. $y_i \neq y_j$. The paths $P_1 = Px_{i-1} \circ (x_{i-1}, y', x_{j+1}) \circ x_{j+1}P$ and $P_2 = (y_i, x_i) \circ x_iPx_j \circ (x_j, y_j)$.

- $y_t \neq u$;
- $y_{t+1} \neq v$.

Proof. We know that $y_t \neq y_{t+1}$ because P is zigzag-free. Then, the first condition follows trivially. Assume that $y_t = u$. Then $y_{t+1} \neq v$, since P is zigzag-free. Now, consider the paths $P_1 = Px_t \circ (x_t, y_t = u, v)$ and $P_2 = (y_{t+1}, x_{t+1}) \circ x_{t+1}P$. Paths P_1 and P_2 meet conditions (i) through (iv). Finally, assume that $y_{t+1} = v$. Then $y_t \neq u$, since P is zigzag-free. Consider the two paths $P_1 = Px_t \circ (x_t, y_t)$ and $P_2 = (u, v = y_{t+1}, x_{t+1}) \circ x_{t+1}P$. Paths P_1 and P_2 meet conditions (i) through (iv). □

By Claim 1, we conclude that $y_j \neq y_{j+1}$, $y_j \neq u$ and $y_{j+1} \neq v$. By Lemma 4 there is a vertex $y \in Y$ such that $(y, x) \in A(D)$ for some $x \in V(P)$. Let $(y_i, x_i) \in A(D)$ such that i is minimum. We shall now show that $i \leq j$. Let $S = \{x_j, x_{j+1}\}$. Since $y_{j+1} \neq v$, we know that $(v, x_{j+1}) \notin A(D)$. If (a) holds for S, then there is a vertex $y \in Y$ adjacent to both vertices in S. Thus, $(y, x_{j+1}) \in A(D)$ by the choice of the arc a_j and since P is zigzag-free, $(y, x_j) \in A(D)$. If (b) holds for the subset S, then by the choice of the arc a_j, $(x_{j+1}, v) \notin A(D)$ and since $v \neq y_{j+1}$, we deduce that $(v, x_{j+1}) \notin A(D)$. Therefore, u and v are adjacent to x_j where $(u, x_j) \in A(D)$ as $y_j \neq u$. Since i is minimum, the analysis of these two cases allow us to conclude $i \leq j$. Moreover, $i > 1$, otherwise P is not zigzag-free. Then, vertex x_{i-1} exists and so does arc (x_{i-1}, y_{i-1}) by the minimality of i and Lemma 5. Again, by Claim 1, we conclude that $y_{i-1} \neq y_i$, $y_{i-1} \neq u$ and $y_i \neq v$.

To conclude the base case, let $S' = \{x_{i-1}, x_{j+1}\}$. Since $y_{j+1} \neq v$ and $y_{i-1} \neq u$, neither x_{j+1} nor x_{i-1} can be simultaneously adjacent to u and v; therefore, (b) cannot hold for S'. Then, (a) holds for S' and there is a vertex y' such that it is adjacent to both vertices of S'. By the choice of j, we have that $(y', x_{j+1}) \in A(D)$. Therefore, $y' \neq y_j$ because P is zigzag-free. By a symmetric reasoning we deduce that $(x_{i-1}, y') \in A(D)$ and $y' \neq y_i$. By Claim 1 we have that $y' \neq u$ and $y' \neq v$.

Vertices y_i and y_j may or may not be the same. Consider first the case in which $y_i \neq y_j$. Then, paths $P_1 = Px_{i-1} \circ (x_{i-1}, y', x_{j+1}) \circ x_{j+1}P$ and $P_2 = (y_i, x_i) \circ x_iPx_j \circ (x_j, y_j)$ satisfy conditions (i) through (iv) (Figure 1).

We may thus assume that $y_i = y_j$. We will now show that some vertex x_t, $i \leq t \leq j$, is not adjacent to y'. To do so, assume the contrary. Then, by Lemma 1, since $(y', x_{j+1}) \in A(D)$, we must have arc $(y', x_{i-1}) \in A(D)$ as well; a contradiction to the

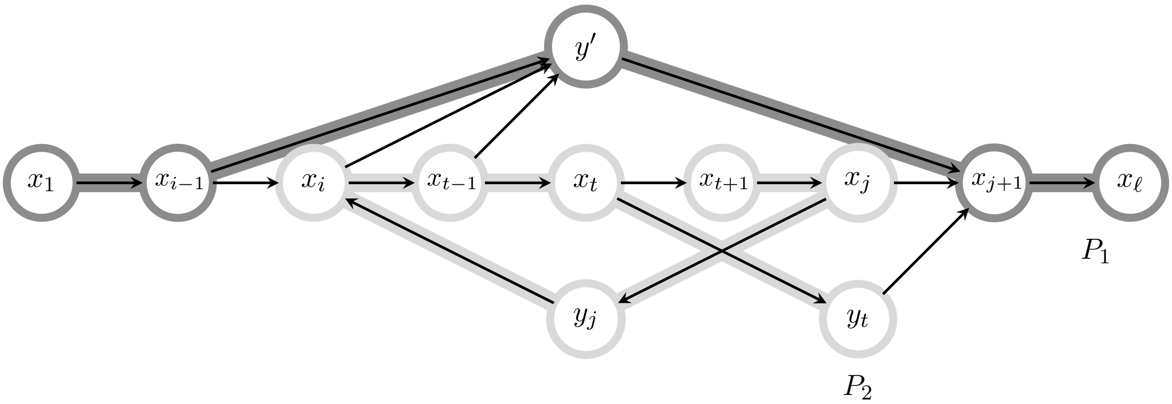

Figure 2. $y_i = y_j$ and $(x_t, y_t) \in A(D)$. The paths $P_1 = Px_{t-1} \circ (x_{t-1}, y', x_{j+1}) \circ x_{j+1}P$ and $P_2 = (y_t, x_t) \circ x_t P x_j \circ (x_j, y_j)$ satisfy the conditions (i) through (iv).

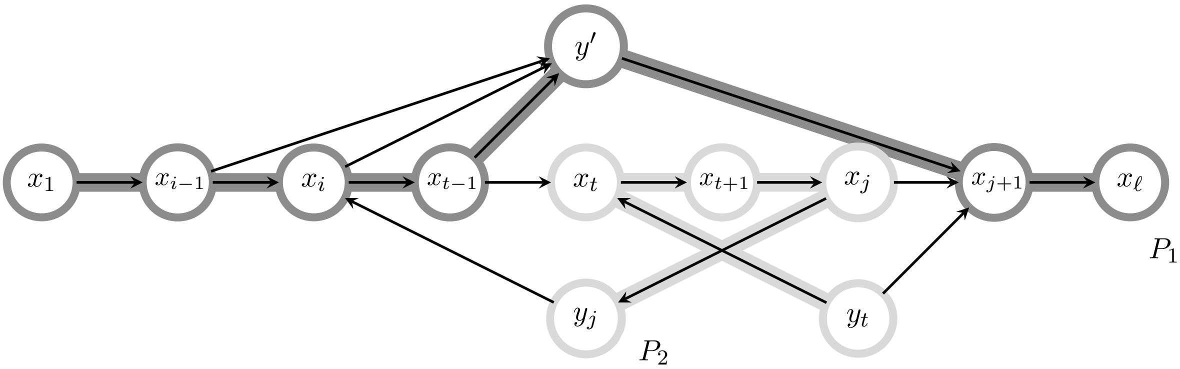

Figure 3. $y_i = y_j$ and $(y_t, x_t) \in A(D)$. The paths $P_1 = Px_{t-1} \circ (x_{t-1}, y', x_{j+1}) \circ x_{j+1}P$ and $P_2 = (y_t, x_t) \circ x_t P x_j \circ (x_j, y_j)$ satisfy the conditions (i) through (iv).

choice of i. Thus, choose $i \leq t \leq j$ such that t is minimum and x_t is not adjacent to y'. By Lemma 5, there is some vertex $y_t \in Y$ adjacent to x_t. As y_t is adjacent to x_t, clearly $y_t \neq y'$. We shall now show that y_t is distinct from $y_i = y_j$. Assume to the contrary that $y_t = y_i = y_j$ is the only vertex in Y adjacent to x_t. Recall that $y_i \neq u$ and $y_j \neq v$. Let $S_t = \{x_t, x_{j+1}\}$. Since $y_i \neq u$ and $y_j \neq v$, (a) must hold for S_t. Therefore, $(x_j, y_j = y_t) \in A(D)$ and $(y_t, x_{j+1}) \in A(D)$; so P has a zigzag, a contradiction. We thus deduce that y_t is distinct from $y_i = y_j$.

If $(x_t, y_t) \in A(D)$, then the paths $P_1 = Px_{i-1} \circ (x_{i-1}, y', x_{j+1}) \circ x_{j+1}P$ and $P_2 = x_{t+1}Px_j \circ (x_j, y_j = y_i, x_i) \circ x_i P x_t \circ (x_t, y_t)$ satisfy the conditions (i) through (iv) (Figure 2).

If $(y_t, x_t) \in A(D)$ note that, by the choice of t we know that y' is adjacent to every vertex in $(x_{i-1}, \ldots, x_{t-1})$. Moreover, since $(x_{i-1}, y') \in A(D)$, by Lemma 1 $(x_{t-1}, y') \in A(D)$. Then, the paths $P_1 = Px_{t-1} \circ (x_{t-1}, y', x_{j+1}) \circ x_{j+1}P$ and $P_2 = (y_t, x_t) \circ x_t P x_j \circ (x_j, y_j)$ satisfy the conditions (i) through (iv) (Figure 3).

Finally, the proof of the base case $k = 2$ is complete. Assume thus that $k > 2$.

By Lemma 3 there is some vertex $x_j \in V(P)$ such that there is some arc $(x_j, y_j) \in A(D)$ for $y_j \in Y$ and $j \geq k - 1$. Among all such arcs choose an arc a_j with j maximum. Since P is zigzag-free, we know that $j < \ell$ and thus there is a vertex x_{j+1} in P. Let

$X' = V(Px_j)$ and let

$$Y' = \{y' : y' \in Y \text{ and } y' \text{ is adjacent to } x_{j+1}\}.$$

Let $D' = D[X', Y']$ and let $P' = Px_j$. If Y' is a stable set, then D' is a spine digraph and according to [Sambinelli et al. 2017, Lemma 3], D' has paths P'_1 and P'_2 that meet the conditions (i) through (iv) above. We may thus assume that Y' contains u and v.

We shall show that P' is zigzag-free in D'. Assume the contrary. Then, since P is zigzag-free in D, this implies that there either is an arc $(x_{j-1}, u) \in A(D')$ or $(x_j, y) \in A(D')$ for some vertex $y \in Y'$. However, if $(x_{j-1}, u) \in A(D')$, since $v \in Y'$ then $(v, x_{j+1}) \in A(D)$ and P would not be zigzag-free in D, a contradiction. Similarly, if $(x_j, y) \in A(D')$ for some vertex $y \in Y'$, since $(y, x_{j+1}) \in A(D)$, then P would not be zigzag-free in D, again a contradiction.

We shall now show that D' is a $(k-1)$-tight arc-spine digraph. Let $S' \subseteq X'$ be an arbitrary set of $k-1$ vertices; by Lemma 3 we know that $j \geq k-1$. Let $S = S' \cup \{x_{j+1}\}$ be a set of k vertices of D. Since D is k-tight, either (a) or (b) holds for S. If (a) holds for S in D, then it is easy to see that (a) also holds for S' in D'. So suppose that (b) holds for S in D. We do know that $\{u, v\} \in Y'$, therefore, $(u, x_{j+1}) \in A(D')$ and $(v, x_{j+1}) \in A(D')$. So, the only vertex in S not adjacent to u and v is some vertex of S' and (b) holds for S' in D'.

We have thus shown that D' is a $(k-1)$-tight arc-spine digraph and P' is zigzag-free. By the induction hypothesis applied to D' and P', there exist paths P'_1 and P'_2 in D' which satisfy conditions (i) through (iv). Assume, without loss of generality, that $ter(P'_1) = x_j$ and $ter(P'_2) = y'$, for some $y' \in Y'$. Let $P_1 = P'_1 \circ (x_j, y_j)$ and $P_2 = P'_2 \circ (y', x_{j+1}) \circ x_{j+1}P$ be paths of D. We claim that P_1 and P_2 meet conditions (i) through (iv). Conditions (iii) and (iv) obviously hold. Condition (i) holds because P'_1 and P'_2 are disjoint by induction hypothesis and neither vertex y nor any vertex of $x_{j+1}P$ are vertices of D'. Condition (ii) holds because $|P'_1| + |P'_2| = j + k$ by induction hypothesis. Therefore

$$|P_1| + |P_2| = |P'_1| + |P'_2| + |X| - j + 1 = |X| + k + 1$$

and the proof is complete. □

Theorem 1. *Let $D[X, Y]$ be a arc-spine digraph. Then, $\pi_k(D) \leq \alpha_k(D)$.*

Proof. We may assume that D is k-tight, otherwise the result follows by Lemma 2. We know that $\alpha_k(D) \geq |Y| + \min\{|X|, k-1\}$. Since D is k-tight, we have by definition that $|X| \geq k$ and, therefore, that $\alpha_k(D) \geq |Y| + k - 1$. Now, suppose that $\lambda(D) > |X|$. Since $\lambda(D) > |X|$, there exists a path P in D such that $|P| = |X| + 1$. Let $\mathcal{P} = P \cup \{(v) : v \notin V(P)\}$. Clearly, $\mathcal{P}$ is a path partition of D and $|\mathcal{P}|_k = \min\{|P|, k\} + |Y| - 1 = |Y| + k - 1$. Therefore, $\pi_k(D) \leq |\mathcal{P}|_k = |Y| + k - 1 \leq \alpha_k(D)$ and the result follows. Hence, we may assume that $\lambda(D) = |X|$. Let $P = (x_1, x_2, \ldots, x_l)$ be a Hamilton path in $D[X]$. Since $\lambda(D) = |X|$, we have that P is a longest path in D; as such, it must be zigzag-free. By Lemma 6, there exist disjoint paths P_1 and P_2 in D such that $|P_1| + |P_2| = |X| + k + 1$. Note that $|P_i| > k$, for $i = 1, 2$, otherwise P_{3-i} would be larger than $|X|$. Let $\mathcal{P} = \{P_1, P_2\} \cup \{(y) : y \notin V(P_1) \cup V(P_2)\}$. It is easy to see that $\mathcal{P}$ is a path partition in D. The k-norm of $\mathcal{P}$ is $|\mathcal{P}|_k = \min\{|P_1|, k\} + \min\{|P_2|, k\} + |Y| - k - 1 = |Y| + k - 1$. So, $\pi_k(D) \leq |Y| + k - 1$ and the result follows. □

3. Conclusion

Sambinelli, Nunes da Silva and Lee adapted the technique of the proof of Linial's Dual Conjecture for split digraphs to prove Linial's Conjecture for spine digraphs. In this paper we were able to apply the same technique to a superclass of spine digraphs. The most important statement proved is Lemma 6, whose assertion and structure of the inductive proof has similar elements to that of [Sambinelli et al. 2017, Lemma 3]. Even though spine and arc-spine digraphs are very similar in structure, the proof of the base case for arc-spine digraphs happens to be a lot more complex than the base case for spine digraphs. It is hard to understand at this moment what does that represent. Intuition suggests that it might be possible to adapt the structure of the proof presented here to deal with superclasses of spine digraphs more complex in structure than arc-spine digraphs.

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